M266. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 17.
2. The sum of the two numbers is 89.
3. The product of the four digits is 210.
4. The product of the two numbers is 1924.

Determine the two numbers.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Since $1924 = 2^2 \cdot 13 \cdot 37$, there are exactly 12 positive divisors of 1924, namely 1, 2, 4, 13, 26, 37, 52, 74, 148, 481, 962, and 1924. Thus, the only factorizations of 1924 as the product of a pair of two-digit numbers are $26 \cdot 74$ and $37 \cdot 52$. It follows from any one of the first three properties that the numbers we seek are 37 and 52.

Remark: If we assume only properties 2 and 4, we need not make any assumption about the digits of the numbers: if two numbers x and y satisfy $x + y = 89$ and $xy = 1924$, it follows that the numbers are 37 and 52.

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; D. KIPP JOHNSON, Beaverton, OR, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; VEDULA N. MURTY, Dover, PA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M267. *Proposed by the Mayhem Staff.*

Find a quintic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 7 repeated k times, then $f(n)$ consists of the digit 7 repeated $5k+3$ times. (For example, $f(77) = 7777777777777$.) Compare with M256 [2006 : 265].

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Let $f(x)$ be a polynomial with the desired property. If n is a positive integer consisting of the digit 7 repeated k times, then $n = \frac{7}{9}(10^k - 1)$. We require $f(n)$ to consist of the digit 7 repeated $5k + 3$ times; that is, $f(n) = \frac{7}{9}(10^{5k+3} - 1)$.

Since $n = \frac{7}{9}(10^k - 1)$, we have $10^k - 1 = \frac{9}{7}n$, and thus $10^k = \frac{9}{7}n + 1$. Then

$$f(n) \;=\; \tfrac{7}{9}(1000 \cdot (10^k)^5 - 1) \;=\; \tfrac{7}{9}\left(1000 \cdot \left(\tfrac{9}{7}n + 1\right)^5 - 1\right) .$$

Thus, $f(x) = \frac{7}{9}\left(1000 \cdot \left(\frac{9}{7}x + 1\right)^5 - 1\right)$.

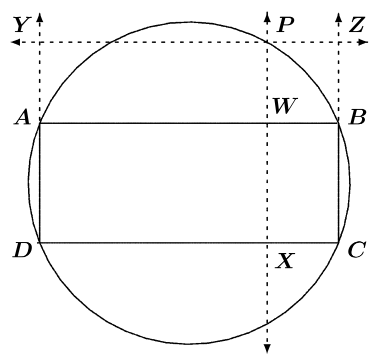
We can write $f(x)$ in a somewhat nicer form:

$$\begin{aligned} f(x) &= \tfrac{7000}{9}\left(\tfrac{9}{7}x+1\right)^5 - \tfrac{7}{9} \;=\; \tfrac{7000}{9}\left(\left(\tfrac{9}{7}x+1\right)^5 - 1\right) + \tfrac{7000}{9} - \tfrac{7}{9} \\ &= 777 + \tfrac{7000}{9}\left(\left(\tfrac{9}{7}x+1\right)^5 - 1\right)\,. \end{aligned}$$

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and D. KIPP JOHNSON, Beaverton, OR, USA.

M268. *Proposed by the Mayhem Staff.*

Rectangle $ABCD$ is inscribed in a circle Γ and P is a point on Γ. Lines parallel to the sides of the rectangle are drawn through P and meet one pair of sides at points W and X and the extensions of the other pair of sides at Y and Z. Prove that the line through W and Y is perpendicular to the line through X and Z.



Similar solutions by Courtis G. Chryssostomos, Larissa, Greece; Hasan Denker, Istanbul, Turkey; and Vedula N. Murty, Dover, PA, USA.

If P is coincident with any of the four points A, B, C, or D, then the statement is true.

Suppose now that $P \notin \{A, B, C, D\}$. Let the centre of circle Γ be at $(0,0)$, and let the coordinates of the vertices of the rectangle inscribed in Γ be $A(-a,b)$, $B(a,b)$, $C(a,-b)$, and $D(-a,-b)$, for some $a > 0$ and $b > 0$. Let P have coordinates (x_0, y_0). The coordinates of points W, X, Y, and Z are then $W(x_0,b)$, $X(x_0,-b)$, $Y(-a,y_0)$, and $Z(a,y_0)$.

The equation of Γ is $x^2 + y^2 = a^2 + b^2$. Since $P(x_0,y_0)$ is on Γ, we have $x_0^2 + y_0^2 = a^2 + b^2$; that is,

$$\frac{x_0^2 - a^2}{y_0^2 - b^2} \;=\; -1\,. \tag{1}$$

The slope of the line through W and Y is $\dfrac{y_0 - b}{-a - x_0}$, and the slope of the line through X and Z is $\dfrac{y_0 + b}{a - x_0}$. The product of these two slopes is

$$\left(\frac{y_0 - b}{-a - x_0}\right)\cdot\left(\frac{y_0 + b}{a - x_0}\right) \;=\; \frac{y_0^2 - b^2}{x_0^2 - a^2} \;=\; -1\,,$$

where the last step uses (1). Thus, the lines WY and XZ are perpendicular.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.